

RANDOM OPERATORS AND CROSSED PRODUCTS

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ABSTRACT. This article is concerned with crossed products and their applications to random operators. We study the von Neumann algebra of a dynamical system using the underlying Hilbert algebra structure. This gives a particularly easy way to introduce a trace on this von Neumann algebra. We review several formulas for this trace, show how it comes as an application of Connes' non commutative integration theory and discuss Shubin's trace formula. We then restrict ourselves to the case of an action of a group on a group and include new proofs for some theorems of Bellissard and Testard on an analogue of the classical Plancherel Theorem. We show that the integrated density of states is a spectral measure in the periodic case, thereby generalizing a result of Kaminker and Xia. Finally, we discuss duality results and apply a method of Gordon et al. to establish a duality result for crossed products by \mathbb{Z} .

0. Introduction

Families of random operators arise in the study of disordered media. More precisely, one is given a topological space X and a family of operators $(H_x)_{x \in X}$ on $L^2(G)$. Here, X represents the set of "all manifestations" of a fixed kind of disorder on the locally compact abelian group G [3,4].

The simplest example of a disordered medium is given by the periodic structure of a crystal. In this case X is the quotient of G by the subgroup of periods. In the general case X will not be a quotient of G , but there will still be an action α of G on X . The fact that all points of X stem from the same kind of disorder structure is taken account of by requiring the action to be ergodic.

Whereas for a fixed $x \in X$ the operator H_x may not have a large symmetry group, the whole family of operators will be G -invariant. This leads to the study of this family as a new object of interest. This study is best performed in the context of C^* -algebras. In fact, it turns out that the crossed products $G \times_{\alpha} C(X)$ provide a natural framework for these objects [3,6,7,10,36].

As it is, one is even led to a more general algebraic structure, viz C^* -algebras of groupoids when studying certain quasicrystals modelled by tilings [21,26,27]. But this is not considered here.

In a remarkable series of papers Bellissard and his Co-workers introduced a K-theory based method called "gap labelling" for the study of random operators [3,4,5,19].

Using results of Pimsner and Voiculescu [32], they were able to get a description of the possible gaps in many important cases.

As K-theory is best known in the cases, where either G is discrete or X stems from an almost periodic function, much of their work was devoted to these cases. However, there are many important examples of more general random operators [11,12,28,29].

This is one of the starting points of this article. In fact, the main purpose of Sections 1 and 2 is to study the framework of general random operators. This is done by means of Hilbert algebras. Sections 3 and 4 are then devoted to special results in the field of random operators. More precisely, this article is organized as follows.

In Section 1 we introduce crossed products, study two important representations and revise their basic theory. Special attention is paid to the relationship between symmetry properties of random operators and direct integral decompositions.

In Section 2 we use Hilbert algebras to the study of the von Neumann algebras and the C^* -algebras of the dynamical systems of Section 1. We use them to introduce a trace on these von Neumann algebras. We show that this trace coincides with the trace introduced by Shubin for almost periodic operators [36] and with the trace studied by Bellissard and others for discrete G [3,4,5]. Moreover, we discuss how it is connected with Connes' non commutative integration theory [13].

In Section 3 we study the case that X is a group itself. We study the relation between the two representations introduced in Section 1. We provide proofs for some theorems first announced in [6] and [7] (cf. [2] as well), whose proofs never seem to have appeared in print. Moreover, we revise the Bloch theory for periodic operators from the algebaric point of view. This point of view has the advantage that the operators in questions are neither required to have pure point spectrum nor to have a kernel. We show that the integrated density of states is a spectral measure in this case for purely algebraic reasons. This generalizes a result of Kaminker and Xia [25] and simplifies their proof.

Finally, in Section 4, we adapt a method developped by Gordon et al. [22] for the study of the almost Mathieu operator to general crossed products by \mathbb{Z} .

1. The C^* -algebra $G \times_{\alpha} C_0(X)$

To every dynamical system (G, α, X) a C^* -algebra can be constructed called the crossed product of G and $C_0(X)$ and denoted by $G \times_{\alpha} C_0(X)$. If X consists of only one point, then $G \times_{\alpha} C_0(X)$ is nothing but the group C^* -algebra $C^*(G)$. We will be concerned with two special representations of $G \times_{\alpha} C_0(X)$. For further details on general crossed products we refer to [31,38], for details on topological dynamics and crossed products see [39,40,41].

1.1 Basic Definitions. A dynamical system is a triple (G, α, X) consisting of

- a separable, metrizable, locally compact, abelian group G , whose Haar measure will be denoted by ds ,
- a separable, metrizable space X ,
- a continuous action α of G on X ,

Moreover we will need

- an α -invariant measure on X with $supp m = X$

to define the representations considered below. We emphasize that this measure is not needed to define the crossed product.

The group G is acting on $L^2(G) := L^2(G, ds)$ by

$$T_t : L^2(G) \rightarrow L^2(G), \quad T_t \xi(s) := \xi(s - t), \quad s, t \in G, \xi \in L^2(G)$$

and on $L^2(X) := L^2(X, dm)$ by

$$S_t : L^2(X) \rightarrow L^2(X), \quad S_t \xi(x) := \xi(\alpha_{(-t)}(x)), \quad t \in G, x \in X, \xi \in L^2(X).$$

Given a topological space Y , we denote by $C_c(Y)$ ($C_0(Y)$, $C_b(Y)$ resp.) the algebra of continuous functions with compact support (vanishing at infinity, being bounded resp). Let $\|\cdot\|_\infty$ denote the supremum norm on either of these algebras. The crossed product $G \times_\alpha C_0(X)$ is defined in the following way:

Equipped with multiplication, involution and norm defined by

$$\begin{aligned} a * b(t, x) &:= \int_G a(s, x) b(t - s, \alpha_{(-s)}(x)) ds, \\ a^*(t, x) &:= \bar{a}(-t, \alpha_{(-t)}(x)), \\ \|a\|_1 &:= \int_G \|a(s, \cdot)\|_\infty ds, \\ a, b &\in C_c(G \times X), t \in G, x \in X, \end{aligned}$$

$C_c(G \times X)$ becomes a normed $*$ -algebra. In general, this algebra is neither complete nor a C^* -algebra. It is easy to see that

$$\|a\| := \sup\{\|\rho(a)\| : \rho \text{ Hilbert space representation of } C_c(G \times X)\}$$

defines a C^* -seminorm on $C_c(G \times X)$. In fact $\|\cdot\|$ is a norm, as can be seen by using the representations π_x , $x \in X$, to be defined below. The crossed product $G \times_\alpha C_0(X)$ of the dynamical system (G, α, X) is the completion of $C_c(G \times X)$ with respect to $\|\cdot\|$. It is immediate from these definitions that every representation of $(C_c(G \times X), \|\cdot\|_1)$ has a unique continuous extension to a representation of $G \times_\alpha C_0(X)$. We will be concerned with two special representations and their direct integral decompositions.

REMARK 1. The algebra $\mathcal{A} := (C_c(G \times X), *, \langle \cdot | \cdot \rangle)$ where involution and convolution are defined as above and $\langle \cdot | \cdot \rangle$ is the inner product on the Hilbert space $L^2(G \times X)$ can easily be seen to be a Hilbert algebra (cf. [18]) i.e. to fulfil the following conditions:

- (i) $\langle a | b \rangle = \langle b^* | a^* \rangle$ $a, b \in \mathcal{A}$;
- (ii) $\langle a * b | c \rangle = \langle b | a^* * c \rangle$ $a, b, c \in \mathcal{A}$;
- (iii) For $a \in \mathcal{A}$ the mapping $b \mapsto a * b$ is continuous;
- (iv) $\{a * b : a, b \in \mathcal{A}\}$ is dense in $L^2(G \times X)$.

In particular, the action of \mathcal{A} on itself from the left yields a representation of \mathcal{A} on $L^2(G \times X)$, which can be extended to a representation of $G \times_\alpha C_0(X)$ on $L^2(G \times X)$. These considerations will be given a more precise form in Section 2 in order to study $G \times_\alpha L^\infty(X)$. Now we prefer to introduce two representations, that allow a direct integral decomposition.

1.2 Representations of $G \times_\alpha C_0(X)$.

Let $\pi : C_c(G \times X) \longrightarrow B(L^2(G \times X))$ be given by

$$\pi(a)\xi(t, x) := \int_G a(t - s, \alpha_t(x)) \xi(s, x) ds, \quad \xi \in L^2(G \times X),$$

and let for $x \in X$ the representation $\pi_x : C_c(G \times X) \longrightarrow B(L^2(G))$ be given by

$$\pi_x(a)\xi(t) := \int_G a(t-s, \alpha_t(x))\xi(s) ds, \quad \xi \in L^2(G),$$

where $B(H)$ denotes the algebra of bounded operators on the Hilbert space H . Then it is easy to see that π and π_x , $x \in X$, are continuous representations of $C_c(G \times X)$. Their extensions to $G \times_\alpha C_0(X)$ will also be denoted by π and π_x . Identifying $L^2(G \times X)$ with $\int_X^\oplus L^2(G) dm$, we get

$$\pi(A) = \int_X^\oplus \pi_x(A) dm, \quad A \in G \times_\alpha C_0(X).$$

This is obvious for $A \in C_c(G \times X)$ and follows for arbitrary $A \in G \times_\alpha C_0(X)$ by density. As G is amenable, even abelian, the representation π is faithful (cf. 7.7 in [31]). Therefore we have

$$G \times_\alpha C_0(X) \simeq \pi(G \times_\alpha C_0(X)) = \overline{\pi(C_c(G \times X))}.$$

Thus the crossed product is the norm closure of an algebra of certain integral operators.

We remark that the mapping $X \ni x \mapsto \pi_x(A)$ is strongly continuous for $A \in G \times_\alpha C_0(X)$, as can be directly calculated for $A \in C_c(G \times X)$, and then follows by density arguments for all $A \in G \times_\alpha C_0(X)$. The representation π has two interesting symmetry properties, that will be given in the next proposition.

PROPOSITION 1.2.1. (a) *Let \mathcal{D} be the algebra of diagonalisable operators on the Hilbert space $L^2(G \times X) \simeq \int_X^\oplus L^2(G) dm$, then*

$$\pi(G \times_\alpha C_0(X)) \subset \mathcal{D}'.$$

(b) *$\pi(G \times_\alpha C_0(X)) \subset \{T_t \otimes S_{-t} : t \in G\}'$. Moreover for $t \in G$, $x \in X$ and $A \in G \times_\alpha C_0(X)$ the formula $T_t \pi_{\alpha_t(x)}(A) T_t^* = \pi_x(A)$ holds.*

PROOF. (a) This is the fact that $\pi(A)$ permits a direct integral decomposition.

(b) This can be directly calculated for $A \in C_c(G \times X)$ and then follows for arbitrary $A \in G \times_\alpha C_0(X)$ by a density argument. \square

Moreover, the following is valid.

PROPOSITION 1.2.2.. *If α is minimal i.e. $Gx := \{\alpha_t(x) : t \in G\}$ is dense in X for every $x \in X$ then*

(i) *$\sigma(\pi_x(A))$ is independent of $x \in X$ for selfadjoint $A \in G \times_\alpha C_0(X)$,*

(ii) *π_x is faithful for every $x \in X$.*

If G acts ergodically on X , then there exists for each selfadjoint $A \in G \times_\alpha C_0(X)$ a closed set $\Sigma_A \subset \mathbb{R}$ and a measurable set $X_A \subset X$ with $\mu(X - X_A) = 0$ s.t. $\sigma(A_x) = \Sigma_A$ for all $x \in X_A$.

PROOF. The first statement is proven in [25], the second one in Section 4 of [3].

\square

We will now give a second representation of $G \times_\alpha C_0(X)$. For $\hat{t} \in \hat{G}$ let $\pi^{\hat{t}} : C_c(G \times X) \rightarrow B(L^2(X))$ be defined by

$$\pi^{\hat{t}}\xi(x) := \int_G a(s, x)\xi(\alpha_{-s}(x))(\hat{t} | -s)ds, \quad a \in C_c(G \times X), \quad \xi \in L^2(X),$$

where $(\cdot | \cdot)$ denotes the dual pairing between G and \hat{G} . For $\hat{t} \in \hat{G}$ the mapping $\pi^{\hat{t}}$ is then a representation of $C_c(G \times X)$, which has a unique continuous extension to a representation of $G \times_\alpha C_0(X)$, again denoted by $\pi^{\hat{t}}$. For $A \in G \times_\alpha C_0(X)$ the mapping $\hat{t} \mapsto \pi^{\hat{t}}(A)$ is strongly continuous, as can be seen using the same arguments as in the case of the mapping $x \mapsto \pi_x(A)$. Therefore, we can define a representation

$$\tilde{\pi} := \int_{\hat{G}}^{\oplus} \pi^{\hat{t}} d\hat{t} : G \times_\alpha C_0(X) \rightarrow B\left(\int_{\hat{G}}^{\oplus} L^2(X) d\hat{t}\right),$$

where we denote by $d\hat{t}$ the Haar measure on \hat{G} . Let the unitaries W and U be defined by

$$W : L^2(G \times X) \longrightarrow L^2(G \times X), \quad W\xi(t, x) := \xi(t, \alpha_t(x)), \quad \xi \in L^2(G \times X),$$

$$U := (F_G \otimes I)W^* : L^2(G \times X) \longrightarrow L^2(\hat{G} \times X),$$

where $F_G : L^2(G) \longrightarrow L^2(\hat{G})$ is the Fourier transform and I the identity. Then we have

$$\tilde{\pi} = U \pi U^*,$$

where $L^2(\hat{G} \times X)$ is identified with $\int_{\hat{G}}^{\oplus} L^2(X) d\hat{t}$.

REMARKS 1. The crossed product $G \times_\alpha C_0(X)$ is just the group C^* -algebra $C^*(G)$, if X consists of only one point. In this case we identify $C_c(G \times X)$ with $C_c(G)$ and we get $\pi(\varphi)\xi = \varphi * \xi = T_\varphi\xi$, $\tilde{\pi}(\varphi)\xi = M(F(\varphi))\xi$, where T_φ denotes the operator of convolution with $\varphi \in C_c(G)$ and $M(\psi)$ denotes the operator of multiplication with ψ . This implies $\tilde{\pi}(G \times_\alpha C_0(X)) = M(C_0(\hat{G}))$ and $\pi(G \times_\alpha C_0(X)) = \{F^{-1}M_\psi F \mid \psi \in C_0(\hat{G})\}$.

2. The direct integral decomposition of $\tilde{\pi}$ relies essentially on the symmetry

$$\pi(G \times_\alpha C_0(X)) \subset \{T_t \otimes S_{-t} : t \in G\}',$$

as can be seen in the following way: Using $W(T_t \otimes I)W^* = T_t \otimes S_{-t}$, one gets immediately $U(T_t \otimes S_{-t})U^* = M_t \otimes I$, where M_t denotes the operator of multiplication with $(t | \cdot)$ on $L^2(\hat{G})$. Therefore, we have $U\pi(A)U^* \in \{M_t \otimes I \mid t \in G\}'$, and this implies (cf. 5, ch 2, II in [18]) that $U\pi(A)U^*$ has a direct integral decomposition.

2. The von Neumann algebra $G \times_\alpha L^\infty(X)$

In this section we study the von Neumann algebra $G \times_\alpha L^\infty(X)$. We will be particularly interested in determining its generators and its commutant as well as introducing and calculating a trace on it. In a sense, much more general considerations can be found in [23], where arbitrary von Neumann crossed products are studied by means of Tomita Takesaki theory of left Hilbert algebras (cf. [38]). The trace on $G \times_\alpha L^\infty(X)$ allows to introduce for each selfadjoint operator affiliated to $G \times_\alpha L^\infty(X)$ a canonical spectral measure. This spectral measure is called the *density of states*. We will discuss the so called *Shubins's trace formula*, relating the density of states to the number of eigenvalues of certain restricted operators. We conclude the section with the discussion of certain formulas for the trace in the case that $m(X) < \infty$ holds.

2.1 Definition and important properties of $G \times_\alpha L^\infty(X)$.

Following 7.10.1 in [31] we define the von Neumann crossed product.

DEFINITION 2.1.1. $G \times_\alpha L^\infty(X) := \pi(G \times_\alpha C_0(X))''$

We will study this algebra by means of the already defined Hilbert algebra $\mathcal{A} = (C_c(G \times X), *, *, \langle \cdot | \cdot \rangle)$ (cf. Remark 1 in Section 1.1). We need some notation.

DEFINITION 2.1.2. (a) For $A \in \mathcal{A}$ let L_a (R_a) be the unique continuous operator with

$$L_a \xi = a * \xi, \quad (R_a \xi = \xi * a), \quad \xi \in C_c(G \times X),$$

i.e. $L_a \xi = \int_G a(s, x) \xi(t - s, \alpha_{-s}(x)) ds$ for $\xi \in L^2(G \times X)$ and similarly for R_a .

(b) The unique extension of $*$: $C_c(G \times X) \rightarrow C_c(G \times X)$ to a continuous mapping of $L^2(G \times X)$ into itself will also be denoted by $*$, i.e. $a^*(t, x) = \overline{a(-t, \alpha_{-t}(x))}$ for $a \in L^2(G \times X)$.

(c) An $a \in L^2(G \times X)$ is called left bounded (right bounded) if the mapping $\xi \mapsto R_\xi a$ ($\xi \mapsto L_\xi a$) can be extended to a continuous operator on $L^2(G \times X)$. This operator will be denoted by L_a (R_a).

(d) $\mathcal{L}(\mathcal{A}) := \{L_a : a \in \mathcal{A}\}''$, $\mathcal{R}(\mathcal{A}) := \{R_a : a \in \mathcal{A}\}''$.

The connection between these crossed products and Hilbert algebras is simple.

LEMMA 2.1.3. For $a \in C_c(G \times X)$ the equality $W^* \pi(a) W = L_a$ holds.

PROOF. For $\xi \in C_c(G \times X)$ a direct computation yields $W^* \pi(a) W \xi = L_a \xi$ and the lemma follows, as L_a and $W^* \pi(a) W$ are bounded and $C_c(G \times X)$ is dense in $L^2(G \times X)$. \square

The lemma and the definitions of $G \times_\alpha L^\infty(X)$ and $\mathcal{L}(\mathcal{A})$ directly yield

THEOREM 2.1.4. $Ad_W : G \times_\alpha L^\infty(X) \rightarrow \mathcal{L}(\mathcal{A})$, $A \mapsto W^* A W$ is a spatial isomorphism of von Neumann algebras.

Those operators which are inverse images of left bounded operators under Ad_W will play an important role.

DEFINITION 2.1.5. (a) A function $a \in L^2(G \times X)$ is called the kernel of the operator $A \in G \times_\alpha L^\infty(X)$ if a is left bounded and $A = W L_a W^*$.

(b) Let $\mathcal{K} := \{A \in G \times_\alpha L^\infty(X) \mid A \text{ has a kernel}\}$.

We study \mathcal{K} in the next proposition.

PROPOSITION 2.1.6. (a) The operator A has the kernel $a \in L^2(G \times X)$ iff

$$A \xi(t, x) = \int_G a(t - s, \alpha_t(x)) \xi(s, x) ds \quad a.e.$$

holds for $\xi \in L^2(G \times X)$.

(b) The set \mathcal{K} is an ideal in $G \times_\alpha L^\infty(X)$. For $A \in \mathcal{K}$ with kernel $a \in L^2(G \times X)$ and $B \in G \times_\alpha L^\infty(X)$ the kernel of AB is given by $W^* B(Wa)$ and the kernel of A^* is given by a^* .

(c) For $A \in G \times_\alpha L^\infty(X)$ with kernel $a \in L^2(G \times X)$ the operator A_x is a bounded Carleman operator with kernel

$$a_x(t, s) := \overline{a(t - s, \alpha_t(x))}$$

(i.e. $(A_x)f(t) = \langle a_x(t, \cdot) | f \rangle$ a.e. $t \in G$ for a.e. $x \in X$).

PROOF. (a) The statement with " $\xi \in L^2(G \times X)$ " replaced by " $\xi \in C_c(G \times X)$ " is easy to calculate. Using that the maximal operator given by the integral expression is closed, we get (a).

(b) As \mathcal{K} contains the algebra $\pi(C_c(G \times X))$ by Lemma 2.1.3., it is strongly dense in $G \times_\alpha L^\infty(X)$. The remaining statements could be calculated directly but also follow from Proposition 2 and Proposition 3 in 3, ch. 5, I of [18].

(c) We set $a_x(t, s) := \overline{a(t - s, \alpha_t(x))}$ for $a \in L^2(G \times X)$. Using the Fubini theorem it is easy to see that a_x is the kernel of a Carleman operator (cf. [42]) \widetilde{A}_x for almost every $x \in X$. It remains to show $\widetilde{A}_x = A_x$, a.e. $x \in X$. For $\eta \in L^2(X)$ and $\xi \in L^2(G)$ a short calculation yields

$$\eta(x)A_x\xi(t) = \eta(x)\widetilde{A}_x\xi(t), \text{ a.e.}$$

As $\eta \in L^2(G)$ was arbitrary this implies

$$A_x\xi = \widetilde{A}_x\xi, \text{ a.e. } x \in X.$$

Using a countable, dense subset of $\xi \in L^2(G)$ and the fact that \widetilde{A}_x is closed, we conclude (c). \square

We can now characterize $G \times_\alpha L^\infty(X)$ and its commutant.

THEOREM 2.1.7. (a) $G \times_\alpha L^\infty(X) = W\mathcal{L}(\mathcal{A})W^* = \{T_t \otimes I, W(I \otimes M_v)W^* : t \in G, v \in L^\infty(X)\}''$.

(b) $G \times_{\tilde{\alpha}} L^\infty(X) = \mathcal{R}(\mathcal{A}) = \{T_t \otimes I, W^*(I \otimes M_v)W : t \in G, v \in L^\infty(X)\}''$ with $\tilde{\alpha} : G \times X \rightarrow X$, $\tilde{\alpha}_t(x) := \alpha_{-t}(x)$.

(c) $(G \times_\alpha L^\infty(X))' = W\mathcal{R}(\mathcal{A})W^* = \{T_t \otimes S_{-t}, I \otimes M_v : t \in G, v \in L^\infty(X)\}''$

PROOF. (a) The equality $G \times_\alpha L^\infty(X) = W\mathcal{L}(\mathcal{A})W^*$ has already been proven in Theorem 2.1.4. To prove the second equality we set

$$\mathcal{C} := \{T_t \otimes I, W(I \otimes M_v)W^* : t \in G, v \in L^\infty(X)\}.$$

$\mathcal{C}'' \subset W\mathcal{L}(\mathcal{A})W^*$: By $\mathcal{L}(\mathcal{A}) = \mathcal{R}(\mathcal{A})'$ (cf. Theorem 1 in 2, ch 5, I of [18]), it is enough to show $\mathcal{C} \subset (W\mathcal{R}(\mathcal{A})W^*)'$, i.e.

$$CWR_aW^* = WR_aW^*C, \quad a \in C_c(G \times X), C \in \mathcal{C}.$$

This can be calculated directly.

$W\mathcal{L}(\mathcal{A})W^* \subset \mathcal{C}''$: For $u \in C_c(G)$ and $v \in C_c(X)$ and $\xi \in L^2(G \times X)$ an easy calculation yields

$$\pi(u \otimes v)\xi(t, x) = (W(I \otimes M_v)W^*\xi(t, x)) \cdot \int_G u(s)(I \otimes T_s)\xi(t, x)ds,$$

and this implies $\pi(u \otimes v) \subset \mathcal{C}''$. The desired inclusion follows.

(b) Defining $\pi_{\tilde{\alpha}}$ by simply replacing α by $\tilde{\alpha}$ in the definition of π , we get

$$R_a\xi = \pi_{\tilde{\alpha}}(Wa)\xi, \quad a \in C_c(G \times X), \xi \in L^2(G \times X).$$

This implies $\mathcal{R}(\mathcal{A}) = \{\pi_{\tilde{\alpha}}(Wa) : a \in C_c(G \times X)\}'' = \{\pi_{\tilde{\alpha}}(a) : a \in C_c(G \times X)\}''$. As $G \times_{\tilde{\alpha}} L^\infty(X) = \{\pi_{\tilde{\alpha}}(a) : a \in C_c(G \times X)\}''$ by definition of the crossed product the first equality is proven. The second equality follows by replacing α by $\tilde{\alpha}$ in (a), i.e. by replacing W by W^* .

(c) The first equality follows from (a) and $\mathcal{R}(\mathcal{A}) = \mathcal{L}(\mathcal{A})'$. The second equality follows by (b) and $W(T_t \otimes I)W^* = T_t \otimes S_{-t}$. \square

Theorem 2.1.7 yields $G \times_\alpha L^\infty(X) \subset D'$. In particular (cf. 5, ch,2, II in [18]), every $A \in G \times_\alpha L^\infty(X)$ can be written as a direct integral $A = \int_X^\oplus A_x dm$, whose fibres are uniquely determined up to a set of measure zero. Similarly it can be seen that for $A \in G \times_\alpha L^\infty(X)$ the equation

$$UAU^* = \int_{\hat{G}}^\oplus A^{\hat{t}} d\hat{t}$$

holds, where the $A^{\hat{t}}$ are uniquely determined up to a set of measure zero. From now on we will identify $G \times_\alpha C_0(X)$ with $\pi(G \times_\alpha C_0(X))$. For A in $G \times_\alpha C_0(X)$ we set $A_x := \pi_x(A)$, $A^{\hat{t}} := \pi^{\hat{t}}(A)$ and $\hat{A} := UAU^*$. For $A \in G \times_\alpha L^\infty(X)$ we define the A_x and $A^{\hat{t}}$ by

$$A = \int_X^\oplus A_x dm \text{ and } UAU^* = \int_{\hat{G}}^\oplus A^{\hat{t}} d\hat{t}.$$

The fact that these families are only defined up to a set of measure zero will be no inconvenience.

REMARK 1. It is always possible to choose the A_x such that

$$T_t^* A_{\alpha_t(x)} T_t = A_x$$

holds for all $x \in X$ and all $t \in G$. This can be seen in the following way: Theorem 2.1.7 implies $G \times_\alpha L^\infty(X) \subset \{T_t \otimes S_{-t} \mid t \in G\}'$. In particular, we have for fixed $t \in G$

$$T_t A_{\alpha_t(x)} T_t^* = A_x, \text{ a.e. } x \in X.$$

Therefore we get, using the Fubini Theorem, that the family of operators defined by

$$\langle \widetilde{A}_x \xi \mid \eta \rangle := M(t \mapsto \langle T_t A_{\alpha_t x} T_t^* \xi \mid \eta \rangle),$$

where M is the mean on the abelian group G , coincides almost everywhere with the family A_x . Moreover, it is easy to see that the family \widetilde{A}_x has the required invariance property.

2.2 The trace on $G \times_\alpha L^\infty(X)$.

In the last section we proved that $G \times_\alpha L^\infty(X)$ is generated by a Hilbert algebra. This allows to introduce a canonical trace on $G \times_\alpha L^\infty(X)$.

We start with a simple lemma that will allow us to prove the equality of certain weights by proving the equality of the restrictions of these weights to suitable sets.

LEMMA 2.2.1. *Let \mathcal{J} be a strongly dense ideal in a von Neumann algebra $\mathcal{N} \subset B(H)$ containing the identity I of $B(H)$.*

(a) *There is an increasing net I_λ in \mathcal{J} converging strongly towards I . If H is separable, (I_λ) can be chosen as a sequence.*

(b) *If τ_1 and τ_2 are normal weights on \mathcal{N} with $\tau_1(AA^*) = \tau_2(AA^*) < \infty$ for $A \in \mathcal{J}$, then $\tau_1 = \tau_2$ on $(\mathcal{J}\mathcal{J})^+$.*

(c) *If τ_1 and τ_2 are normal weights on \mathcal{N} , whose restrictions to $(\mathcal{J}\mathcal{J})^+$ coincide, then $\tau_1 = \tau_2$.*

PROOF. (a) The Ideal $\mathcal{J}\mathcal{J}$ is normdense in the C^* algebra $\mathcal{C} := \overline{\mathcal{J}\mathcal{J}}$. By 1.7.2 in [17], there exists therefore an approximate unit I_λ in $\mathcal{J}\mathcal{J}$ for \mathcal{C} . As the net I_λ is bounded and increasing, it converges strongly to some $E \in B(H)$ with

$$EC = CE = C, \quad C \in \mathcal{C}.$$

As \mathcal{J} is strongly dense in \mathcal{N} , the algebra \mathcal{C} is weakly dense in \mathcal{N} and $EC = CE = C$, $C \in \mathcal{N}$, follows. This implies $E = I$. If H is separable it is possible to choose an increasing subsequence of (I_λ) converging to E .

(b) This follows using polarisation.

(c) Let (I_λ) be as in (a). Fix $A = CC^*$ in \mathcal{N}^+ . As τ_1 and τ_2 are normal, it is enough to show

$$\tau_1(CI_\lambda C^*) = \tau_2(CI_\lambda C^*).$$

But this is clear, as $\mathcal{J}\mathcal{J}$ is an ideal and as (I_λ) belongs to $(\mathcal{J}\mathcal{J})^+$. \square

THEOREM 2.2.2. *There exists a unique normal trace τ on $G \times_\alpha L^\infty(X)$ with*

$$\tau(AA^*) = \langle a | a \rangle$$

for A with kernel a . The trace τ is semifinite and normal and $\tau = \tau_c \circ \text{Ad}_W^$, where τ_c is the canonical trace on $\mathcal{L}(\mathcal{A})$ (cf. 2, ch 6, I of [18]). Moreover*

$$(G \times_\alpha L^\infty(X))_\tau^2 := \{A \in G \times_\alpha L^\infty(X) : \tau(AA^*) < \infty\} = \mathcal{K}.$$

PROOF. Clearly the identity of $B(L^2(G \times X))$ is contained in $G \times_\alpha L^\infty(X) = \pi(G \times_\alpha C_0(X))''$ and we can apply the foregoing lemma with $\mathcal{J} = \mathcal{K}$ to get the uniqueness.

As Ad_W is an isomorphism by Theorem 2.1.4., the remaining statements follow easily from the corresponding statements in 2, ch 6, I of [18]. \square

DEFINITION 2.2.3. For a selfadjoint operator A affiliated to the von Neumann algebra $G \times_\alpha L^\infty(X)$, i.e. whose resolution of the identity, E_A [35], is contained in $G \times_\alpha L^\infty(X)$, define

$$\mu_A(B) := \tau(E_A(B))$$

for Borel measurable $B \subset R$. The map μ is called the *integrated density of states* (IDS) for A (cf. [6]).

We mention that there is a different approach to the IDS for one dimensional Schrödinger operators due to Johnson and Moser (cf. [16, 24]).

From Theorem 2.2.2 we get the following corollaries.

COROLLARY 2.2.4. *Let A and μ_A be as in the preceding definition. Then μ_A is a spectral measure for A .*

PROOF. This is clear, as τ is faithful and normal. \square

COROLLARY 2.2.5. *Let A and μ be as in Corollary 2.2.4. If there exists a set $\sigma \subset R$ with $\sigma(A_x) = \sigma$ a.e. $x \in X$, then $\sigma = \text{supp } \mu$.*

PROOF. As μ is a spectral measure of A , we have $\sigma(A) = \text{supp } \mu$. By $\sigma(A_x) = \sigma$ a.e. $x \in X$, the equality $\sigma = \sigma(A)$ holds. \square

There is another way to calculate the trace that can be seen as an application of [13] (cf. Remark 1 below).

LEMMA 2.2.6. *Let A be in $(G \times_\alpha L^\infty(X))_+$. Then there exists a unique $\Lambda(A) \in [0, \infty]$ with*

$$\Lambda(A) \int_G g^2(t) dt = \int_X \text{tr}(M_g A_x M_g) dm$$

for positive $g \in L^\infty(G)$, where tr denotes the usual trace on $B(L^2(G \times X))$.

PROOF. Uniqueness is obvious. Existence will follow from the uniqueness of the Haar measure on G , once we have shown that the RHS of the equation induces an invariant measure.

As A is positive there exists $C \in G \times_\alpha L^\infty(X)$ with $A = C^*C$. We calculate

$$\mu(B) := \int_X \text{tr}(M_{\chi_B} A_x M_{\chi_B}) dm = \int_X \text{tr}(M_{\chi_B} C_x^* C_x M_{\chi_B}) dm.$$

Using that tr is a trace we conclude $\mu(B) = \int_X \text{tr}(C_x M_{\chi_B} C_x^*) dm$.

This formula and some simple monotone convergence arguments show that μ is a measure with

$$\int_G g^2 d\mu = \int_X \text{tr}(C_x M_{g^2} C_x^*) dm = \int_X \text{tr}(M_g A_x M_g) dm.$$

It remains to show that μ is translation invariant. As $G \times_\alpha L^\infty(X)$ is contained in $\{T_t \otimes S_{-t} : t \in G\}'$ for each $t \in G$, the equation

$$C_{\alpha_t(x)} = T_t^* C_x T_t$$

holds for a.e. $x \in X$. This allows to calculate

$$\begin{aligned} \mu(B - t) &= \int_X \text{tr}(C_x M_{\chi_{B-t}} C_x^*) dm \\ &= \int_X \text{tr}(C_x T_t M_{\chi_B} T_t^* C_x^*) dm \\ (\text{tr is trace}) &= \int_X \text{tr}(T_t^* C_x T_t M_{\chi_B} T_t^* C_x^* T_t) dm \\ &= \int_X \text{tr}(C_{\alpha_t(x)} M_{\chi_B} C_{\alpha_t(x)}^*) dm \\ &= \mu(B). \end{aligned}$$

Here we used in the last equation that m is translation invariant. The calculation shows that μ is translation invariant. This finishes the proof. \square

THEOREM 2.2.7. $\Lambda = \tau$.

PROOF. As \mathcal{K} is a strongly dense ideal in $G \times_\alpha L^\infty(X)$ by Proposition 2.1.6., it is by Lemma 2.2.1 enough to show

$$\Lambda(A^*A) = \tau(A^*A)$$

for $A \in \mathcal{K}$. Choosing a positive $g \in L^\infty(G)$ with $\int_G g^2 dt = 1$, we calculate for $A \in \mathcal{K}$ with kernel $a \in L^2(G \times X)$

$$\begin{aligned} \Lambda(A^*A) &= \int_X \text{tr}(M_g A_x^* A_x M_g) dm \\ &= \int_X \int_{G \times G} g(t) a(t-s, \alpha_t(x)) |a(t-s, \alpha_t(x))|^2 dt ds dm \\ (\text{Fubini}) &= \int_G |g(t)|^2 \left(\int_G \int_X |a(t-s, \alpha_t(x))|^2 dm ds \right) dt \\ (m, ds \text{ transl. inv.}) &= \int_G |g(t)|^2 \left(\int_G \int_X |a(s, x)|^2 dm ds \right) dt \\ (\|g\|_{L^2(G)} = 1) &= \langle a | a \rangle, \end{aligned}$$

where we used that for an operator $K \in B(L^2(G))$ with kernel $k \in L^2(G \times G)$ the equality $\text{tr}(KK^*) = \int_{G \times G} |k(t, s)|^2 dt ds$ holds. \square

In some cases (e.g. in the almost periodic case or if $G = R^n, Z^n$) it is known that there exists a sequence $H_n \subset G$ with

$$\lim_{n \rightarrow \infty} \frac{1}{m_G(H_n)} \int_G \chi_{H_n}(s) f(\alpha_s(x)) dm_G(s) = \int_X f(z) dm(z)$$

for $f \in L^1(X, m)$ and $x \in X_f$, where X_f is a suitable subset of X with $\mu(X - X_f) = 0$. If this is valid, and if $A \in \mathcal{K}$ has a kernel a s.t. $x \mapsto \int_G |a(t, x)|^2 dt$ belongs to $L^1(X, m)$, then

$$\lim_{n \rightarrow \infty} \frac{1}{m_G(H_n)} \text{tr}(\chi_{H_n} A_x A_x^* \chi_{H_n}) = \tau(AA^*), \quad (*)$$

by $\text{tr}(\chi_{H_n} A_x A_x^* \chi_{H_n}) = \int_G \int_G \chi_{H_n}(t) |a(s, \alpha_{-t}(x))|^2 ds dt$. Here, one can interpret a term of the form $\chi_H B_x \chi_H$ as the restriction $B_x|_H$ of B_x to $L^2(H, m_G|_H)$. One is in particular interested in the case, where $A_x = \chi_I(B_x)$ belongs to the resolution of the identity of B_x . As restrictions of operators are comparatively accessible, the question arises whether

$$\lim_{n \rightarrow \infty} \frac{1}{m_G(H_n)} (\text{tr}(\chi_{H_n} \chi_I(B_x) \chi_{H_n}) - \text{tr}(\chi_I(B_x|_{H_n}))) = 0$$

for $x \in X$ and $I \subset R$. If this can be established, the equation

$$\mu_A(I) = \lim_{n \rightarrow \infty} \frac{1}{m_G(H_n)} \text{tr}(\chi_I(B_x|_{H_n}))$$

follows from (*) immediately, as the equation $\chi_I(C) = \chi_I(C)\chi_I(C)^*$ holds for arbitrary operators C . Note that $\text{tr}(\chi_I(B_x|_{H_n}))$ is just the number of eigenvalues of $B_x|_{H_n}$ in I . For $I = (-\infty, E]$, $E \in R$, this type of equation has been established for pseudodifferential operators with almost periodic coefficients in [36], for Schrödinger operators in [6] using heat equation methods and for discrete G in [3]. It is called *Shubin's trace formula*.

REMARKS 1. In [13] transverse functions and transverse measures on groupoids are introduced and studied (cf. I, 5 of [14] as well). It is possible to give $G \times X$ the structure of a groupoid. The measure m then induces a unique transversal measure Λ with certain properties. It is possible to show that Λ satisfies the equation

$$\Lambda(\xi) \int_G f dt = \int_X \xi_x(f) dm$$

for transverse functions ξ and $f \in L^\infty(G)$. A direct calculation shows that for $A \in (G \times_\alpha L^\infty(X))_+$ the mapping $\xi^x(B) := \text{tr}(\chi_B A_x \chi_B)$ is a transverse function (if the components A_x are chosen according to Remark 1 in Section 2.1). In this context Theorem 2.1.7 says essentially $\Lambda(\xi_A) = \tau(A)$.

2. In [1] it is shown that for a family A_ω , $\omega \in \Omega$, of almost periodic Schrödinger operators and $F \in C_0(\mathbb{R})$ the equation

$$\int_\Omega \text{tr}(M_g F(A_\omega) M_g) dm_\Omega = \int_{\mathbb{R}} F dk$$

holds, where the measure dk is given by a certain limit procedure. Using Definition 2.2.3 and Theorem 2.2.7 we see $dk = d\mu_A$.

2.3 Some special cases.

If $m(X) < \infty$ (e.g. if X is compact) there exist two alternative formulas for the trace on $G \times_\alpha L^\infty(X)$. They will now be discussed.

Define for $A \in G \times_\alpha L^\infty(X)$ the operator $A_m : L^2(G) \longrightarrow L^2(G)$ by

$$\langle A_m \xi | \eta \rangle := \int_X \langle A_x \xi | \eta \rangle dm, \quad \xi, \eta \in L^2(G).$$

Since $A_{\alpha_t x} = T_t A_x T_t^*$ a.e. $x \in X$ for fixed $t \in G$ and m is invariant under α , the operator A_m is translation invariant. Therefore there exists $\varphi \in L^\infty(\widehat{G})$ with $A_m = F^{-1} M_\varphi F$, where M_φ denotes the operator of multiplication by φ . Now it is easy to see that the mapping

$$J : G \times_\alpha L^\infty(X) \longrightarrow M(L^\infty(\widehat{G})), \quad A \mapsto F A_m F^{-1}$$

is linear, positive and faithful on $(G \times_\alpha L^\infty(X))_+$. Let τ_∞ be the usual trace on $M(L^\infty(\widehat{G}))$, i.e. $\tau_\infty(\varphi) := \int_{\widehat{G}} \varphi d\hat{t}$.

Moreover, define $\mu : (G \times_\alpha L^\infty(X))_+ \longrightarrow [0, \infty]$ by

$$\mu(A) := \int_{\widehat{G}} \langle A^{\hat{t}} 1 | 1 \rangle d\hat{t},$$

where 1 denotes the function of $L^2(\widehat{G})$ with constant value 1. Then the following holds.

THEOREM 2.3.1. $\tau = \mu = \tau_\infty \circ J$.

PROOF. We will show (1) $\tau = \tau_\infty \circ J$ and (2) $\tau = \mu$.

(1) $\tau = \tau_\infty \circ J$: By Theorem 2.2.7, it is enough to show $\Lambda(A) = \tau_\infty \circ J(A)$ for $A \in (G \times_\alpha L^\infty(X))_+$. For such an A let A_m and the function ϕ be defined as above, i.e. $A_m = F^{-1}M_\phi F$ and $M_\phi = J(A)$. Choosing a positive $g \in L^\infty(G)$ with $\|g\|_{L^2(G)} = 1$ we calculate

$$\Lambda(A) = \int_X \text{tr}(M_g A_x M_g) dm = \text{tr}(M_g A_m M_g) = \text{tr}(M_g F^{-1} M_{\phi^{1/2}} F F^{-1} M_{\phi^{1/2}} F M_g).$$

For $\phi \in L^1(\widehat{G})$ the operator $K = M_g F^{-1} M_{\phi^{1/2}} F$ is a Hilbert Schmidt operator with kernel $k(t, x) = g(t) F^{-1}(\phi^{1/2})(t - s)$. Thus the formula $\text{tr}(K^* K) = \int_{G \times G} |k(t, s)|^2 dt ds$ holds and we get

$$\Lambda(A) = \int_G \int_G |g(t) F^{-1}(\phi^{1/2})(t - s)|^2 dt ds = \|\phi^{1/2}\|_{L^2(\widehat{G})}^2 = \int_{\widehat{G}} \phi(\hat{t}) d\hat{t},$$

where we used the translation invariance of dt , $\|g\|_{L^2(G)} = 1$, and the fact that the Fourier transform is an isometry.

For arbitrary ϕ the equality $\Lambda(A) = \int_{\widehat{G}} \phi(\hat{t}) d\hat{t}$ now follows by a simple monotone limit procedure.

(2) $\tau = \mu$: For $A \in (G \times_\alpha L^\infty(X))_+$ let A_m and φ be as above. By (1), it is enough to show $\varphi(\hat{t}) = \langle A^{\hat{t}} 1 | 1 \rangle =: \psi(\hat{t})$ for a.e. $\hat{t} \in \widehat{G}$. But this follows from the following calculation valid for all $\eta, \xi \in L^2(\widehat{G})$:

$$\begin{aligned} \langle M_\varphi \xi | \eta \rangle &= \langle A_m F^{-1} \xi | F^{-1} \eta \rangle \\ &= \int_X \langle A_x F^{-1} \xi | F^{-1} \eta \rangle dm \\ &= \langle A(I \otimes F^{-1})(1 \otimes \xi) | (I \otimes F^{-1})(1 \otimes \eta) \rangle \\ &= \langle AU^*(1 \otimes \xi) | U^*(1 \otimes \eta) \rangle \\ &= \langle \hat{A}(1 \otimes \xi) | (1 \otimes \eta) \rangle \\ &= \langle M_\psi \xi | \eta \rangle \end{aligned}$$

The theorem is proven. \square

REMARKS 1. The expression μ was used in [25] (cf. also [9]).

2. The mapping J was first introduced by Coburn, Moyer and Singer [10] (cf. also [36]) in their paper on almost periodic operators.

3. If X consists of only one point it is a fortiori compact. The positive operators in $G \times_\alpha C_0(X)$ are just the operators $A = F_G^{-1} M_\varphi F_G$ where $\varphi \in C_0(\widehat{G})$ is positive (cf. Remark 1 in Section 1). The trace of such an A is then given by

$$\tau(A) = \int_{\widehat{G}} \varphi(\hat{t}) d\hat{t},$$

in particular $\tau(A)$ is finite iff φ belongs to $L^1(\widehat{G})$. As the ideal of trace class operators consists of the finite linear combinations of positive operators with finite trace, we conclude that $A = F_G^{-1} M_\varphi F_G$ is trace class iff φ belongs to $L^1(\widehat{G}) \cap C_0(\widehat{G})$.

In particular it is not true in general that an operator of the form $A = \pi(\varphi) = F^{-1}M_{F(\varphi)}F$ with $\varphi \in C_c(G \times X) = C_c(G)$ is trace class. This shows that it is not possible (as is sometimes done) to define a trace on $G \times_\alpha L^\infty(X)$ by setting

$$\tau(\pi(\psi)) := \int_X \psi(0, x) dm.$$

We close this section with some remarks on the special case that G is discrete. As in this case the function $\delta_e : G \rightarrow \mathbb{C}$ defined by $\delta_e(t) = 1$ for $t = e$ and $\delta_e(t) = 0$ for $t \neq e$ is positive, bounded with $\|\delta_e\|_{L^2(G)} = 1$, we get easily

$$\tau(A) = \int_X \langle A_x \delta_e | \delta_e \rangle dm.$$

Moreover, it is possible to show that there exists a conditional expectation

$$J : G \times_\alpha L^\infty(X) \rightarrow L^\infty(X)$$

with $\tau = \tau_\infty \circ J$, where τ_∞ is the usual trace on $L^\infty(X)$ (cf. [31] for a thorough discussion of this case).

3. Groups acting on groups

This section is devoted to the study of a group acting on another group. This situation arises in particular in the context of almost periodic functions (cf. [8, 30]) and this is indeed the motivating example. The irrational rotation algebras, which have recieved a lot of attention (cf. [15, 34] and references therein), arise in such a situation. They are used in the treatment of the so called Harper's model [4]. The corresponding operator is just the almost Mathieu operator for $\lambda = 2$ (cf. [22] and references therein for details about the almost Mathieu operator).

3.1 The general case.

We will look at the following situation: Let G and X be locally compact abelian groups, and let $j : G \rightarrow X$ be a group homomorphism. This induces a homomorphism $j^* : \widehat{X} \rightarrow \widehat{G}$, where \widehat{X} (resp. \widehat{G}) denotes the dual group of X (resp. G). Then there is an action α of G on X given by

$$\alpha_t : X \rightarrow X, \quad \alpha_t(x) := x + j(t),$$

and an action of \widehat{X} on \widehat{G} given by

$$\hat{\alpha}_{\hat{x}} : \widehat{G} \rightarrow \widehat{G}, \quad \hat{\alpha}_{\hat{x}}(\hat{t}) := \hat{t} + j^*(\hat{x}).$$

Similarly to the unitaries T_t and S_t resp., acting on $L^2(G)$ and $L^2(X)$ resp. for $t \in G$, there are unitaries $T_{\hat{x}}$ and $S_{\hat{x}}$ defined by

$$\begin{aligned} S_{\hat{x}} : L^2(\widehat{G}) &\rightarrow L^2(\widehat{G}) \quad S_{\hat{x}}\xi(\hat{t}) := \xi(\hat{\alpha}_{(-\hat{x})}(\hat{t})), \\ T_{\hat{x}} : L^2(\widehat{X}) &\rightarrow L^2(\widehat{X}) \quad S_{\hat{x}}\xi(\hat{y}) := \xi(\hat{y} - \hat{x}). \end{aligned}$$

Moreover we define

$$\widehat{W} : L^2(\widehat{X} \times \widehat{G}) \longrightarrow L^2(\widehat{X} \times \widehat{G}) \quad \text{by} \quad \widehat{W}\xi(\hat{x}, \hat{t}) := \xi(\hat{x}, \hat{\alpha}_{\hat{x}}(\hat{t}))$$

and \tilde{U} by

$$\tilde{U} := (F_X \otimes F_G)W^* = (F_X \otimes I)U,$$

where F_X and F_G are the Fourier transform on $L^2(X)$ and $L^2(G)$ resp. and U and W are as defined in the first section.

We will first establish a spatial isomorphism between the von Neumann algebras $G \times_{\alpha} L^{\infty}(X)$ and $\widehat{X} \times_{\hat{\alpha}} L^{\infty}(\widehat{G})$. This can be done quite easily, as the generators of these von Neumann algebras are known explicitly.

We will then provide proofs for some statements first appearing in [6] and [7], that yield much more, namely an isomorphism between $G \times_{\alpha} C_0(X)$ and $\widehat{X} \times_{\hat{\alpha}} C_0(\widehat{G})$. We will need the following propositions.

PROPOSITION 3.1.1. (a) $W^*(T_t \otimes I) = (T_t \otimes S_t)W^*$, $\widehat{W}^*(T_{\hat{x}} \otimes I) = (T_{\hat{x}} \otimes S_{\hat{x}})W^*$.
(b) $W(T_t \otimes I) = (T_t \otimes S_{-t})W$, $\widehat{W}(T_{\hat{x}} \otimes I) = (T_{\hat{x}} \otimes S_{-\hat{x}})W$.
(c) $\tilde{U}W = \widehat{W}^*\tilde{U}$.

PROOF. This can be seen by direct calculation. \square

PROPOSITION 3.1.2. (a) *The von Neumann algebra $G \times_{\alpha} L^{\infty}(X)$ is generated by operators of the form $T_t \otimes I$, $t \in G$, and the operators of multiplication with the functions*

$$(t, x) \mapsto (\hat{x} | \alpha_t(x)) = (\hat{x} | x)(\hat{x} | j(t)), \quad \hat{x} \in \widehat{X}.$$

(b) *The von Neumann algebra $\widehat{X} \times_{\hat{\alpha}} L^{\infty}(\widehat{G})$ is generated by operators of the form $T_{\hat{x}} \otimes I$, $t \in G$, and the operators of multiplication with the functions*

$$(\hat{x}, \hat{t}) \mapsto (t | \hat{\alpha}_{\hat{x}}(\hat{t})) = (t | j^*(x))(t | \hat{t})$$

PROOF. This follows from Theorem 2.1.7 and the well known fact that the characters generate the von Neumann algebra L^{∞} . \square

Now we can prove the spatial isomorphism, mentioned above.

THEOREM 3.1.3. *The unitary $\tilde{U} : L^2(G \times X) \longrightarrow L^2(\widehat{X} \times \widehat{G})$ establishes a spatial isomorphism between $G \times_{\alpha} L^{\infty}(X)$ and $\widehat{X} \times_{\hat{\alpha}} L^{\infty}(\widehat{G})$.*

PROOF. By the foregoing proposition, it is enough to show

$$(1) \quad \tilde{U}(T_t \otimes I)\tilde{U}^* = M((t | j^*(x))(t | \hat{t})),$$

$$(2) \quad \tilde{U}M((\hat{x} | x)(\hat{x} | j(t)))\tilde{U}^* = T_{\hat{x}} \otimes I,$$

where $M(\varphi)$ denotes as usual the operator of multiplication by φ .

(1) Using Proposition 3.1, we can calculate

$$\begin{aligned} \tilde{U}(T_t \otimes I) &= (F_X \otimes F_G)W^*(T_t \otimes I) \\ &= (F_X \otimes F_G)(T_t \otimes S_t)W^* \\ &= (t | \cdot)(t | j^* \cdot)(F_X \otimes F_G)W^*. \end{aligned}$$

(2) This can be seen by similar arguments. \square

As already stated above there is even an isomorphism between the C^* -algebras $G \times_\alpha C_0(X)$ and $\widehat{X} \times_{\hat{\alpha}} C_0(\widehat{G})$. We will establish this isomorphism in two steps. In the first step we will show that the image of an operator with L^2 -kernel under conjugation by \widetilde{U} is also an operator with an L^2 kernel. In the second step we will prove that conjugation by \widetilde{U} is actually an isomorphism between $G \times_\alpha C_0(X)$ and $\widehat{X} \times_{\hat{\alpha}} C_0(\widehat{G})$. We remark that both of these facts have already been stated in [6] and [7], where, however, no proof was given.

LEMMA 3.1.4. *Let $a \in L^2(G \times X)$ be the kernel of a bounded operator A on $L^2(G \times X)$, i.e.*

$$A\xi(t, x) = \int_G a(t - s, \alpha_s(x)) \xi(s, x) ds.$$

Then $\widetilde{A} := \widetilde{U} A \widetilde{U}^$ is a bounded operator on $L^2(\widehat{X} \times \widehat{G})$ with kernel*

$$\hat{a} := (F_X \otimes F_G) W a \in L^2(\widehat{X} \times \widehat{G}),$$

i.e.

$$\widetilde{A}\xi(\hat{x}, \hat{t}) = \int_{\widehat{X}} \hat{a}(\hat{x} - \hat{y}, \hat{\alpha}_{\hat{x}}(\hat{t})) \xi(\hat{y}, \hat{t}) d\hat{y}.$$

PROOF. Let $\hat{\xi} := (F_X \otimes F_G)(\xi)$ be an arbitrary function in $L^2(\widehat{X} \times \widehat{G})$. We calculate

$$\begin{aligned} \widetilde{U} A \widetilde{U}^* \hat{\xi}(\hat{x}, \hat{t}) &= (F_X \otimes F_G) W^* A W \xi(\hat{x}, \hat{t}) \\ &= (F_X \otimes F_G)((t, x) \mapsto \int_G a(t - s, x) W \xi(s, \alpha_{-t}(x)) ds)(\hat{x}, \hat{t}) \\ &= (F_X \otimes F_G)((t, x) \mapsto \int_G (W a)(t - s, \alpha_{t-s}(x)) \xi(s, \alpha_{t-s}(x)) ds)(\hat{x}, \hat{t}). \end{aligned}$$

As for fixed $t \in G$ the mapping $x \mapsto \int_G |(W a)(t - s, \alpha_{t-s}(x)) \xi(s, \alpha_{t-s}(x))| ds$ belongs to $L^1(X)$, this expression equals

$$(I \otimes F_G) \left(\int_X (\hat{x} | - x) \left(\int_G (W a)(t - s, \alpha_{t-s}(x)) \xi(s, \alpha_{t-s}(x)) ds \right) dx \right) (\hat{x}, \hat{t}),$$

which yields after the substitution $(x \mapsto x - j(t - s))$

$$\begin{aligned} \dots &= (F_X \otimes F_G)((t, x) \mapsto ((\hat{x} | - j(\cdot)) W a)(\cdot, x) * \xi(\cdot, x)(t))(\hat{x}, \hat{t}) \\ &= F_X(x \mapsto (I \otimes F_G)(W a)(\hat{t} + j^*(\hat{x}), x) (I \otimes F_G)(\xi)(\hat{t}, x))(\hat{x}) \\ &= (F_X \otimes F_G)(W a)(\hat{t} + j^*(\hat{x}), \cdot) * (F_X \otimes F_G)(\xi)(\hat{t}, \cdot)(\hat{x}) \\ &= \int_X \hat{a}(\hat{x} - \hat{y}, \hat{\alpha}_{\hat{x}}(\hat{t})) \hat{\xi}(\hat{y}, \hat{t}) d\hat{y}. \end{aligned}$$

This proves the lemma. □

THEOREM 3.1.5. *The mapping $Ad_{\tilde{U}} : G \times_{\alpha} C_0(X) \longrightarrow \widehat{X} \times_{\hat{\alpha}} C_0(\widehat{G})$, $A \mapsto \tilde{U}A\tilde{U}^*$ is an isomorphism of C^* -algebras.*

PROOF. We have to show that $Ad_{\tilde{U}}(G \times_{\alpha} C_0(X))$ is contained in $\widehat{X} \times_{\hat{\alpha}} C_0(\widehat{G})$ and that $Ad_{\tilde{U}^*}(\widehat{X} \times_{\hat{\alpha}} C_0(\widehat{G}))$ is a subset of $G \times_{\alpha} C_0(X)$. We show

- (1) There is a dense set $D \subset G \times_{\alpha} C_0(X)$ with $Ad_{\tilde{U}}(D) \subset \widehat{X} \times_{\hat{\alpha}} C_0(\widehat{G})$.
 - (2) There is a dense subset F of $\widehat{X} \times_{\hat{\alpha}} C_0(\widehat{G})$ with $Ad_{\tilde{U}^*}(F) \subset G \times_{\alpha} C_0(X)$.
- (1) Let

$$D := \{\pi(W^*(g \otimes (h_1 * h_2))) \mid g \in C_c(G), h_1, h_2 \in C_c(X)\}.$$

It is easy to see that D is in fact dense in $G \times_{\alpha} C_0(X)$. The lemma yields that for $A = \pi(W^*(g \otimes (h_1 * h_2))) \in D$ the operator $Ad_{\tilde{U}}(A)$ has the kernel

$$\hat{a} := F_G(g) \otimes (F_X(h_1)F_X(h_2)).$$

But, as $F_G(g)$ belongs to $C_0(\widehat{G})$ and $(F_X(h_1)F_X(h_2))$ belongs to $L^1(\widehat{X})$, the function \hat{a} is indeed the kernel of an operator in $G \times_{\alpha} C_0(X)$.

(2) This can be seen by similar arguments. \square

We now provide a proof for another theorem which was already stated (without proof) in [6] and [7].

THEOREM 3.1.6. *Let τ (resp. $\hat{\tau}$) be the trace on $G \times_{\alpha} L^{\infty}(X)$ (resp. $\widehat{X} \times_{\hat{\alpha}} L^{\infty}(\widehat{G})$) defined in the last section. Then the equation*

$$\tau(A) = \hat{\tau}(\tilde{U}A\tilde{U}^*)$$

holds for all $A \in (G \times_{\alpha} L^{\infty}(X))_+$.

PROOF. It is enough to consider the case $A = BB^*$ with $B \in \mathcal{K}$ with kernel b (cf. Lemma 2.2.1). Then the kernel of $\tilde{U}B\tilde{U}^*$ is given by $(F_G \otimes F_X)(Wb)$ and we have

$$\begin{aligned} \hat{\tau}(\tilde{U}A\tilde{U}^*) &= \int_{\widehat{G}} \int_{\widehat{X}} |(F_G \otimes F_X)(Wb)(\hat{x}, \hat{t})|^2 d\hat{x} d\hat{t} \\ &= \int_G \int_X |Wb(t, x)|^2 dt dx \\ &= \int_X \int_G |b(t, x)|^2 dt dx \\ &= \tau(BB^*). \end{aligned}$$

As $A = BB^*$ the theorem is proven. \square

There is an analogue of the classical Plancherel Theorem.

COROLLARY 3.1.7. *The mapping $Ad_{\tilde{U}}$ establishes an isomorphism between the ideals $(G \times_{\alpha} L^{\infty}(X))_{\tau}^2$ and $(\widehat{X} \times_{\hat{\alpha}} L^{\infty}(\widehat{G}))_{\hat{\tau}}^2$ with*

$$\tau(AA^*) = \hat{\tau}(Ad_{\tilde{U}}(A)Ad_{\tilde{U}}(A)^*).$$

PROOF. This follows directly from Theorem 3.1.6. \square

We will now give a short application of the above theory.

3.2 Periodic operators.

Let H be a closed subgroup of a locally compact abelian group G such that $X := G/H$ is compact. Let $p : G \rightarrow X$ be the canonical projection. Then the machinery developed in the last section can be applied with $j = p$. For $A \in G \times_\alpha L^\infty(X)$ we denote in this section by $\widehat{G} \ni \hat{t} \mapsto A^{\hat{t}}$ the family of operators with

$$\int_{\widehat{G}}^{\oplus} A^{\hat{t}} d\hat{t} = \widetilde{U} A \widetilde{U}^*,$$

where \widetilde{U} was defined in the last section.

The operators in $G \times_\alpha L^\infty(X)$ have a very strong invariance property.

PROPOSITION 3.2.1. *For every $A \in G \times_\alpha L^\infty(X)$ there exist unique A_x , $x \in X$, with (i) $A = \int_X^{\oplus} A_x dm$ and (ii) $T_t A_{\alpha_t(x)} T_t^* = A_x$, $x \in X, t \in G$. The same holds for selfadjoint A that are affiliated to $G \times_\alpha L^\infty(X)$.*

PROOF. For $A \in G \times_\alpha L^\infty(X)$ the existence of such A_x has already been shown (cf. Remark 1 in Section 2). The uniqueness follows as p is surjective. For selfadjoint A affiliated to $G \times_\alpha L^\infty(X)$ the uniqueness proof is unchanged. Existence follows by looking at $(A + i)^{-1}$. \square

We have the following theorem.

THEOREM 3.2.2. *Let A be selfadjoint and affiliated to $G \times_\alpha L^\infty(X)$ with resolution of identity E_A and fibres A_x , $x \in X$, chosen according to the preceding proposition. Then the measure μ defined in Definition 2.2.3 is a spectral measure for all $x \in X$.*

PROOF. By Corollary 2.2.4, the measure μ is a spectral measure for A . As all A_x , $x \in X$, are unitarily equivalent by Proposition 3.2.1, the statement follows. \square

REMARKS 1. Kaminker and Xia show in [25] by the use of a spectral duality principle that certain elliptic periodic operators have purely continuous spectra on the complement of the set of discontinuities of $\lambda \mapsto \tau(E_A((-\infty, \lambda]))$. Theorem 3.2.2 shows in particular that this holds for arbitrary periodic operators for purely algebraic reasons.

2. For periodic Schrödinger operators it is possible to show that the spectrum is purely absolutely continuous using some analyticity arguments (cf. [33] and references therein).

We finish this section with a short discussion of another formula for τ . Let $H^\perp \subset \widehat{G}$ be the annihilator of H and let $q : \widehat{G} \rightarrow \widehat{G}/H^\perp$ denote the canonical projection. For $g = f \circ q$ define $\{g\} := f$. Let for $\rho = q(\hat{t})$ the functional $I_\rho : L^\infty(\widehat{G}) \rightarrow \mathbb{R}$ be defined by

$$I_\rho(f) := \int_{H^\perp} f(\hat{t} + h^\perp) dm_{H^\perp} = \sum_{h \in H^\perp} f(\hat{t} + h^\perp),$$

then the desintegration formula

$$\int_{\widehat{G}} f(\hat{t}) d\hat{t} = \int_{\widehat{G}/H^\perp} I_\rho(f) dm_{\widehat{G}/H^\perp}(\rho)$$

holds (cf. [20]).

Let $A \in (G \times_\alpha L^\infty(X))_+$ be given. Identifying $\widehat{(G/H)}$ with H^\perp and using

$$\mathrm{tr} A^{\hat{t}} = \sum_{h \in H^\perp} \langle A^{\hat{t}} \delta_h | \delta_h \rangle = I_{q(\hat{t})}(\hat{s} \mapsto \langle A^{\hat{s}} \delta_e | \delta_e \rangle),$$

we calculate

$$\begin{aligned} \tau(A) &= \int_{\widehat{G}} \langle A^{\hat{t}} \delta_e | \delta_e \rangle d\hat{t} \\ &= \int_{\widehat{G}/H^\perp} I_\rho(\hat{s} \mapsto \langle A^{\hat{s}} \delta_e | \delta_e \rangle) dm_{\widehat{G}/H^\perp}(\rho) \\ &= \int_{\widehat{G}/H^\perp} \{\hat{s} \mapsto \mathrm{tr} A^{\hat{s}}\}(\rho) dm_{\widehat{G}/H^\perp}(\rho). \end{aligned}$$

The RHS of this equation is essentially the integrated density of states defined in ch. XIII of [33] for periodic operators.

4. Spectral duality

By spectral duality we mean a relation between the spectral types of A_x , $x \in X$, and the spectral types of $A^{\hat{t}}$, $\hat{t} \in \widehat{G}$ of the form "If $A^{\hat{t}}$ has pure point spectrum a.e. $\hat{t} \in \widehat{G}$, then $A^{\hat{t}}$ has purely (absolutely) continuous spectrum a.e. $x \in X$."

Theorems of this form have been stated in [7, 9, 25]. We cite the theorem of [25].

THEOREM 4.1. *Let $A \in G \times_\alpha L^\infty(X)$ be selfadjoint with purely continuous spectrum on a Borel set E s.t. $A^{\hat{t}}$ has pure point spectrum on E for almost all $\hat{t} \in \widehat{G}$, then A^x has purely continuous spectrum on E for almost all $x \in X$.*

In [22] another form of duality is proven for the Almost Mathieu Equation. The method developped there can be carried over with only small changes to give

THEOREM 4.2. *Let (Z, α, X, m) be a dynamical system. Let $A \in Z \times_\alpha L^\infty(X)$ be selfadjoint with spectral family E_A s.t. A_x has only pure point spectrum with simple eigenvalues for almost all $x \in X$. Then*

$$\mu(B) := \tau(E_A(B))$$

is a spectral measure for A^η for almost all $\eta \in \widehat{Z} =: S^1$.

PROOF. As μ is a spectral measure for A by Corollary 2.2.4, it is enough to show that there are spectral measures ν^η for A^η not depending on η . This is shown following [22].

By the same method as in [22], it can be shown that there exist measurable functions

$$N_j : X \longrightarrow Z \cup \{\infty\}, \text{ and } \varphi_j^l : X \longrightarrow l^2(Z), \ j \in Z, l \in N,$$

s.t. the set

$$\{\varphi_j^l(x) \mid j \in Z, l = 1, \dots, N_j(x)\}$$

is an orthonormal basis of $l^2(Z)$ consisting of eigenvectors of A_x for almost all $x \in X$ and that the φ_j^l satisfy

$$\varphi_j^l(x) = T_k \varphi_{j-k}^l(\alpha_k(x)), \ k \in Z.$$

In particular we have

$$\langle \varphi_j^l(x) | T_k \varphi_j^l(\alpha_k(x)) \rangle_{L^2(X)} = 0, \quad k \neq 0,$$

as φ_j^l and $T_k \varphi_j^l(\alpha_k(x)) = \varphi_{j+k}^l(x)$ are different members of an ONB.

The simplicity of the eigenvalues is crucial to get this measurable section of eigenfunctions. We will now show (cf. [22])

- (1) Fix $\psi \in L^2(X)$, $F \in C_0(\mathbb{R})$, $j \in Z$, $l \in N$. Let $\xi(z, x) := \psi(x) \varphi_j^l(x)(z)$, $\hat{\xi}_\eta(x) := U\xi(\eta, x)$ and

$$\mu_\eta(F) := \langle \hat{\xi}_\eta | F(A^\eta) \hat{\xi}_\eta \rangle_{L^2(X)}.$$

Then $\mu_\eta(F)$ is independent of η a.e. η .

- (2) μ_η does not depend on η a.e. η .

- (3) There exist ν_η not depending on η s.t. ν_η is a spectral measure for A^η for a.e. η .

By the remarks at the beginning of the proof, the theorem follows from (3).

- (1) It is enough to show $0 = \int_{S^1} (\eta | z) \mu_\eta(F) d\eta$ for all $z \in Z$ with $z \neq 0$. We calculate

$$\begin{aligned} \int_{S^1} (\eta | z) \mu_\eta(F) d\eta &= \langle (I \otimes M_z) U\xi | UF(A)\xi \rangle \\ &= \langle U(T_z \otimes S_z)\xi | UF(A)\xi \rangle \\ &= \langle (T_z \otimes S_z)\xi | F(A)\xi \rangle. \end{aligned}$$

Using that $\varphi_j^l(x)$ is an eigenvector corresponding to the eigenvalue $e_j^l(x)$ say, we get

$$\begin{aligned} \dots &= \int_X F(e_j^l(x)) \psi(\alpha_z(x)) \psi(x) \langle T_z \varphi_j^l(\alpha_z(x)) | \varphi_j^l(x) \rangle dm \\ &= 0, \end{aligned}$$

where we used the relation $\langle T_z \varphi_j^l(\alpha_z(x)) | \varphi_j^l(x) \rangle = 0$.

- (2) As $C_0(\mathbb{R})$ is separable, this follows from (1).

- (3) Let $\{\psi_l\}$ be an ONB of $L^2(X)$. Then $\{\psi_l \otimes \varphi_j^l\}$ is an ONB in $L^2(G \times X)$ and, as U is unitary, it follows that the $\xi_{l,j,m} := U(\psi_l \otimes \varphi_j^l)$ form an ONB in $L^2(\hat{G} \times X)$. Thus the set $\mathcal{T} := \{\xi_{l,j,m}(\eta, \cdot) | l, j, m\}$ is total in $L^2(X)$ for almost all $\eta \in S^1$. (Notice that the set $M_\varphi := \{\eta \in S^1 | \varphi \perp \xi_{l,j,m}(\eta, \cdot) \forall l, j, m\}$ has measure zero for each $\varphi \in L^2(G)$.) Therefore the measures ν^η defined by

$$\nu^\eta(B) := \sum_{l,j,n} \langle \chi_B(A^\eta) \xi_{l,j,m}(\eta, \cdot) | \xi_{l,j,m}(\eta, \cdot) \rangle_{L^2(G)}$$

are spectral measures for almost all $\eta \in S^1$, which do not depend on η by (2). The theorem follows. \square

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